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EXTERIOR DIFFERENTIAL FORMS ON RIEMANNIAN SYMMETRIC SPACES

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Abstract. In the present paper we give a rough classification of exterior differential forms on a Riemannian manifold. We define conformal Killing, closed conformal Killing, coclosed conformal Killing and harmonic forms due to this classification and consider these forms on a Riemannian globally symmetric space and, in particular, on a rank-one Riemannian symmetric space. We prove vanishing theorems for conformal Killing L^2-forms on a Riemannian globally symmetric space of noncompact type. Namely, we prove that every closed or co-closed conformal Killing L^2-form is a parallel form on an arbitrary such manifold. If the volume of it is infinite, then every closed or co-closed conformal Killing L^2-form is identically zero. In addition, we prove vanishing theorems for harmonic forms on some Riemannian globally symmetric spaces of compact type. Namely, we prove that all harmonic one-forms vanish everywhere and every harmonic *r*-form $(r \geq 2)$ is parallel on an arbitrary such manifold. Our proofs are based on the Bochner technique and its generalized version that are most elegant and important analytical methods in differential geometry "in the large".

Keywords: Riemannian symmetric space, conformal Killing L^2-form, harmonic form

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INTRODUCTION

In the present paper we consider conformal Killing, closed conformal Killing, coclosed conformal Killing and harmonic forms which are defined on Riemannian globally symmetric spaces. In particular, we prove vanishing theorems for conformal Killing, closed conformal Killing and coclosed conformal Killing L^2-forms on Riemannian globally symmetric spaces of noncompact type. In addition, we prove vanishing theorems for harmonic forms on some Riemannian globally symmetric spaces of compact type. Our proofs are based on the *Bochner technique* and its generalized version that are most elegant and important analytical methods in differential geometry "in the large".

The results of the present paper were announced at the International Conference "Differential Geometry" organized by the Banach Center from June 18 to June 24, 2017 at Będlewo (Poland) and at the International Conference "Modern Geometry and its Applications" dedicated to the 225th anniversary of the birth of N.I. Lobachevsky and organized by the Kazan Federal University from November 27 to December 3, 2017 at Kazan (Russian Federation).

PRELIMINARIES

More then thirty years ago Bourguignon has investigated (see [1]) the space of natural (with respect to isometric diffeomorphisms) differential operators of order one determined on vector bundle $\Lambda^r M$ of exterior differential *r*-forms and taking their values in the space of homogeneous tensor fields on (M, g).

Bourguignon has proved the existence of *three basis natural operators* of this space, but only the following two D_1 and D_2 of them were recognized. The first operator D_1 is the exterior differential operator $d : C^\infty\Lambda^r M \to C^\infty\Lambda^{r+1} M$ and the second operator D_2 is the exterior co-differential operator $d^* : C^\infty\Lambda^r M \to C^\infty\Lambda^{r-1} M$.

About the third basis natural operator D_3, it was said that except for case *r* = 1, this operator does not have any simple geometric interpretation. Next, for the case *r* = 1, it was explained that the kernel of this operator consists of *infinitesimal conformal transformations* on (M, g).

In connection with this, we have received a specification of the Bourguignon proposition and proved (see [4]) that the basis of natural differential operators consists of three operators of following forms:

$$D_1 = \tfrac{1}{r+1} d; \quad D_2 = \tfrac{1}{n-r+1} g \wedge d^*;$$
$$D_3 = \nabla - \tfrac{1}{r+1} d - \tfrac{1}{n-r+1} g \wedge d^*$$

where

$$(g \wedge d^*\omega)(X_0, X_1, \dots, X_r) =$$
$$= \sum_{a=2}^{r} (-1)^a g(X_0, X_a)(d^*\omega)(X_1, \dots, X_{a-1}, X_{a+1}, \dots, X_r)$$

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for an arbitrary exterior differential r-form ω and any vector fields $X_1, X_2, ..., X_r$ on M.

The kernel of D_1 consists of *closed exterior differential r-forms*, the kernel of D_2 consists of *co-closed exterior differential r-forms* and kernel of D_3 consists of *conformal Killing r-forms* or, in other words, *conformal Killing-Yano tensors* of order r that constitute three vector spaces $\mathbf{D}^r(M, \mathbb{R})$, $\mathbf{F}^r(M, \mathbb{R})$ and $\mathbf{T}^r(M, \mathbb{R})$ respectively. These vector spaces is subspaces of the vector space of exterior differential r-forms on (M, g) which we denote by $\mathbf{\Omega}^r(M, \mathbb{R})$.

Remark. The concept of conformal Killing tensors was introduced by S. Tachibana about forty years ago (see [5]). He was the first who has generalized some results of a *conformal Killing vector field* (or, in other words, an infinitesimal conformal transformation) to a skew symmetric covariant tensor of order 2 named him the conformal Killing tensor. Kashiwada has generalized this concept to conformal Killing forms of order $r \geq 2$ (see [18]). The theory of conformal Killing forms is contained in the monographs [9] and [15]. In addition, there are many various applications of these tensors in theoretical physics (see, for example, [3]; [4]; [9]; [10]; [20]; [21]; [22]). In particular, we have proved that the vector space $\mathbf{T}^r(M, \mathbb{R})$ on a compact n-dimensional $(1 \leq r \leq n - 1)$ Riemannian manifold (M,g) is finite-dimensional. In addition, the number $\mathrm{t}_r(M) = \mathbf{T}^r(M, \mathbb{R})$ is a conformal invariant M of (M,g) such that $\mathrm{t}_r(M) = \mathrm{t}_{n-r}(M)$.

The condition $\omega \in \ker D_3 \cap \ker D_2$ characterizes an r-form ω as *co-closed conformal Killing form*. Therefore, the space of all co-closed conformal Killing r-forms we can define as $\mathbf{K}^r(M, \mathbb{R}) = \mathbf{T}^r(M, \mathbb{R}) \cap \mathbf{F}^r(M, \mathbb{R})$. A co-closed conformal Killing form is also called as the *Killing-Yano tensor* (see, for example, [9, pp. 426–427]; [10; pp. 559–564]).

In turn, the condition $\omega \in \ker D_1 \cap \ker D_2$ characterizes the form ω as a *closed conformal Killing form* or *closed conformal Killing-Yano tensor* (see, for example, [9, pp. 416]; [23]). Sometimes, closed conformal Killing forms are also called *planar forms* (see [16]). Therefore, the space of all closed conformal Killing r-forms we can define as $\mathbf{P}^r(M, \mathbb{R}) = \mathbf{T}^r(M, \mathbb{R}) \cap \mathbf{D}^r(M, \mathbb{R})$.

Remark. The concept of Killing tensors was introduced by K. Yano about fifteen years ago (see [19]). He was the first who has generalized some results of a *Killing vector field* (or, in other words, an infinitesimal isometric transformation). In turn, planar forms generalized the concept of *concircular vector fields*. In addition, we have proved that the vector spaces $\mathbf{K}^r(M, \mathbb{R})$ and $\mathbf{P}^r(M, \mathbb{R})$ on a compact n-dimensional $(1 \leq r \leq n - 1)$ Riemannian manifold (M,g) are finite-dimensional. Moreover, the numbers $\mathrm{k}_r(M) = \mathbf{K}^r(M, \mathbb{R})$ and $\mathrm{p}_r(M) = \mathbf{P}^r(M, \mathbb{R})$ are conformal invariant of (M,g) such that $\mathrm{k}_r(M) = \mathrm{p}_{n-r}(M)$ (see [15] and [16]).

The condition $\omega \in \ker D_1 \cap \ker D_2$ characterizes the form ω as a *harmonic form* (see [19]; [24, pp. 107–113]). Hence, the space of all harmonic r-forms we can define as $\mathbf{H}^r(M, \mathbb{R}) = \mathbf{D}^r(M, \mathbb{R}) \cap \mathbf{F}^r(M, \mathbb{R})$. On a closed and oriented Riemannian manifold (M, g) the condition $\omega \in \ker d \cap \ker d^*$ is equivalent to the following condition $\omega \in \ker \Delta$ (see, for example, [37 pp. 202]).

Remark. Harmonic forms are a classical object of investigation of differential geometry for the last seventy years, beginning with the well-known scientific works of Hodge (see, for example, [24, pp. 107–113] and [25]). It is well known by Hodge that the vector space $\mathbf{H}^r(M, \mathbb{R})$ of harmonic r-forms on a compact n-dimensional $(1 \leq r \leq n - 1)$ Riemannian manifold (M,g) is finite-dimensional. Moreover, the dimension of $\mathbf{H}^r(M, \mathbb{R})$ equals to the Betti number $\mathrm{b}_r(M)$ of (M,g) such that $\mathrm{b}_r(M) = \mathrm{b}_{n-r}(M)$ (see [2, pp. 202–208; 385–391]).

In conclusion, we denote by $\mathbf{C}^r(M, \mathbb{R})$ the vector space of *parallel* or *covariantly constant r-forms* on M with respect to ∇, i.e. $\mathbf{C}^r(M, \mathbb{R}) = \mathbf{T}^r(M, \mathbb{R}) \cap \mathbf{D}^r(M, \mathbb{R}) \cap \mathbf{F}^r(M, \mathbb{R})$.

Riemannian symmetric spaces are also a classical object of investigation of differential geometry as differential forms. Cartan obtained the basic theory of symmetric spaces between 1914 and 1927. Riemannian symmetric spaces have been studied by many others authors. In particular, beginning in the 1950s, Harish-Chandra, Helgason, and others developed harmonic analysis and representation theory on these spaces and their Lie groups of isometries (see, for example, [2, pp. 235–264]; [13]; [17]; [26]; [38]; [39, pp. 222–292]).

We recall here that the Riemannian symmetric space is a finite dimensional Riemannian manifold (M,g), such that for every its point x there is an involutive geodesic symmetric s_x, such that x is an isolated fixed point of s_x. (M,g) is said to be *Riemannian locally symmetric* if its geodesic symmetries are in fact isometric. This is equivalent to the vanishing of the covariant derivative of the curvature tensor R of (M,g) (see [39, p. 244]). A Riemannian locally symmetric space is said to be a *Riemannian globally symmetric space* if, in addition, its geodesic symmetries are defined on all (M,g). A Riemannian globally symmetric space is complete (see [39, p. 244]). In addition, a complete and simply connected Riemannian locally symmetric space is a Riemannian globally symmetric space (see [39, p. 244]).

Riemannian globally symmetric spaces can be classified by classifying their isometry groups. The classification distinguishes three basic types of Riemannian globally symmetric spaces: spaces of so-called *compact type*, spaces of so-called *non-compact type* and spaces of *Euclidean type* (see, for example, [26; p. 207–208]; [39, p. 252]). An addition, if (M,g) is a Riemannian globally symmetric spaces of compact type then (M,g) is a compact Riemannian manifold with non-negative sectional curvature and positive-definite Ricci tensor (see [39, p. 256]).

Let (M,g) be a Riemannian locally symmetric space, then $\nabla Ric = 0$ for the Ricci tensor Ric of (M,g). If, in addition, (M,g) is irreducible, then it is Einstein (see

[2, p. 254]). If in this case, the Einstein constant is positive, then it follows from Myer's theorem (see [2, p. 171]) that the space (M,g) is compact. Then one can show that the curvature operator is nonnegative (see also [2, p. 254]). Therefore, (M,g) is a Riemannian globally symmetric space of compact type. If, in addition, (M,g) is simply-connected and its curvature operator is positive-definite, then it is a Euclidian sphere (see also [2, p. 228]). Therefore, we can conclude that a simply-connected and irreducible Riemannian locally symmetric space with positive-definite curvature operator is a Euclidian sphere.

On the other hand, if (M,g) is a Riemannian globally symmetric spaces of non-compact type then (M,g) is a complete non-compact Riemannian manifold with non-positive sectional curvature and negative-definite Ricci tensor, and diffeomorphic to a Euclidean space (see [39, p. 256]).

We know that an irreducible Riemannian locally symmetric space (M,g) is Einstein (see [2, p. 254]). If the Einstein constant is negative, then it follows from Bochner's theorem on Killing fields that the space is noncompact (see [2, p. 254]). In this case, one can show that the curvature operator is nonpositive (see also [2, p. 254]). Therefore, (M,g) is a Riemannian globally symmetric space of noncompact type.

CONFORMAL KILLING FORMS ON RIEMANNIAN GLOBALLY SYMMETRIC SPACES

In this section we consider the natural *Hilbert space* $L^2(\Lambda^r M)\cap C^\infty(\Lambda^r M)$ of L^2-forms on a complete noncompact Riemannian manifold (M,g) which is determined by the condition

$$\int_M \|\omega\|^2 d\,Vol_g < \infty$$

for an arbitrary form $\omega \subset C^\infty(\Lambda^r M)$.

First, we prove the following theorem for closed and coclosed conformal Killing L^2-forms on a Riemannian symmetric space of noncompact type that is a complete and simply connected Riemannian manifold of non-positive sectional curvature. Side by side, its curvature operator $\bar{\mathrm{R}}$ is nonpositive (see [39]).

Theorem 1. *Let (M,g) be an n-dimensional ($n \geq 3$) simply connected Riemannian symmetric space of non-compact type. Then every closed or co-closed conformal Killing L^2-form on (M,g) is a parallel form. If the volume of (M,g) is infinite, then every closed or co-closed conformal Killing L^2-form on (M,g) is identically zero.*

Proof. In our paper [31] we have proved that an arbitrary closed (resp. coclosed) conformal Killing L^2-form ω is parallel on a complete noncompact Riemannian manifold (M,g) with nonpositive curvature operator. It means that $\mathbf{C}^r(M, \mathbb{R}) = \mathbf{P}^r(M, \mathbb{R}) = \mathbf{K}^r(M, \mathbb{R})$. In this case, $\|\omega\|^2 = \mathrm{const}$. If we suppose that the volume $Vol_g(M)$ of (M,g) is infinite then we obtain a contradiction with our condition that $\omega \in L^2(\Lambda^r M)$. It remains to recall that a Riemannian globally symmetric spaces of non-compact type (M,g) is a complete non-compact Riemannian manifold with non-positive sectional curvature.

If a Riemannian symmetric space of non-compact type has an even dimension then the following theorem on conformal Killing L^2-forms is true.

Theorem 2. *Let (M,g) be a $2r$-dimensional $(m \geq 2)$ simply connected Riemannian symmetric space of non-compact type. Then every conformal Killing L^2-form of degree r is parallel form on (M,g). If the volume of (M,g) is infinite, then every conformal Killing L^2-form of degree r is identically zero on (M,g).*

Proof. In our paper [31] we have proved that an arbitrary conformal Killing L^2-form of degree r is parallel form on $2r$- dimensional complete non-compact Riemannian manifold (M,g) with negative semi-define curvature operator. It means that $\mathbf{C}^r(M, \mathbb{R}) = \mathbf{T}^r(M, \mathbb{R})$. If the volume of (M,g) is infinite, then every conformal Killing L^2-form of degree r is identically zero on (M,g). In this case, Theorem 2 is a corollary of this proposition.

An important invariant of a symmetric space is its *rank* which is the maximal dimension of a totally geodesic flat subspace. In particular, *rank-one symmetric spaces* are an important class among symmetric spaces (see [16] and [17]). In his book [17], Chavel gave a beautiful account of the rank-one symmetric spaces from a geometric point of view up to the classification of them, which he left for the reader to pursue as a matter in Lie group theory. We prove the following theorem for conformal Killing forms on real rank-one symmetric spaces using his results.

Theorem 3. *Let (M,g) be a simply connected symmetric space of rank one type. If, in addition, (M,g) is a space of compact type, and of odd dimension*
$n = 2k+1$ for $k \geq 1$,
then $t_r(M) = (n+2)!/(r+1)!(n-r+1)!$,
$k_r(M) = (n+1)!/(r+1)!(n-r)!$
and $p_r(M) = (n+1)!/\mathrm{r}!(n-r+1)!$
for an arbitrary $1 \leq r \leq 2k$.

Proof. We consider a rank-one Riemannian symmetric space (M,g) of compact type. If, in addition, (M,g) is a simply connected manifold of odd dimension $n = 2k+1$ then (M,g) is a sphere of constant sectional curvature (see [13]). The vector spaces over $\mathbb{R}$ of conformal Killing, coclosed and closed conformal Killing p-forms on an Euclidian n-sphere have finite dimensions which are equal to
$\mathrm{t_r}(M) = (n+2)!/(r+1)!(n-r+1)!$,
$k_r(M) = (n+1)!/(r+1)!(n-r)!$
and $p_r(M) = (n+1)!/\mathrm{r}!(n-r+1)!$, respectively (see [15] and [16]). This proves our Theorem 3.

HARMONIC FORMS ON A RIEMANNIAN GLOBALLY SYMMETRIC SPACES

In this section we consider harmonic forms on a Riemannian globally symmetric space of compact type. In the simply connected case, "compact type" is equivalent to the compactness condition that was considered in Section 4 in [41].

First we prove the following theorem for harmonic forms on a Riemannian symmetric space of compact type that is a compact Riemannian manifold with non-negative sectional curvature and positive-definite Ricci tensor. Side by side, its curvature operator $\bar{R}$ is nonnegative (see [35]). In this case, the *Bochner technique* tells us that all harmonic r-forms ($2 \le r \le n-1$) are parallel and 1-forms are identically zero (see [2, pp. 208; 212; 221]). Now, a parallel form is necessarily invariant under the holonomy. Thus, we are left with a classical invariance problem (see [37, pp. 306-307]). In this case, the Betti numbers $b_1(M) = b_{n-1}(M) = \dim\ \mathbf{H}^1(M,\ \mathbb{R}) = 0$ and $b_r(M) = \dim\ \mathbf{H}^r(M,\ \mathbb{R}) \le \binom{n}{r}$ for any $r = 2,\ldots,n-2$ (see [2, p. 212]). Since there is exactly one harmonic 0-form (a constant) on compact manifolds and the dual n-form then the Betti numbers $b_0(M) = b_n(M) = 1$. We proved the following theorem.

Theorem 4. *Let* $(M,\ g)$ *be an n-dimensional* ($n \ge 2$) *non-compact simply connected Riemannian globally symmetric space of compact type. Then all harmonic one-forms vanish everywhere and every harmonic r-form* ($r \ge 2$) *is parallel. In this case, the Betti numbers* $b_1(M) = b_{n-1}(M) = 0$ *and* $b_r(M) \le \binom{n}{r}$ *for an arbitrary* $r = 2,\ldots,n-2$.

In conclusion, we can formulate an obvious statement.

Theorem 5. *Let* (M,g) *be an n-dimensional simply connected symmetric space of rank one. If, in addition,* (M,g) *is a manifolds of compact type, and of odd dimension, then its Betti numbers* $b_0(M) = b_n(M) = 1$, $b_1(M) = \ldots = b_{n-1}(M) = 0$. *On the other hand, if* (M,g) *is a simply connected symmetric space of rank one, and of non-compact type, then* (M,g) *carries no harmonic* L^2*-forms except when* $r = n/2$ *in which case* $\mathbf{H}^r(M,\ \mathbb{R})$ *is infinite dimensional.*

Proof. Chavel has proved in [17] that if (M,g) is a rank-one Riemannian symmetric space of compact type then it is one of the four spaces: S^n, $\mathbb{R}P^n$, $\mathbf{C}P^n$, $\mathbf{H}P^n$ and $\mathbf{O}P^2$. The final is the 16-dimensional Cayley plane. Accordingly, compact-type symmetric spaces of rank-one have strictly positive sectional curvature. In the case of odd dimension (M,g) is a Euclidian sphere (see [16]). On the other hand, it is well known that a compact Riemannian manifold (M,g) with positive constant sectional curvature admits no nonzero harmonic forms (see [2, p. 212]). Therefore, its Betti numbers $b_1(M) = \ldots = b_{n-1}(M) = 0$ and $b_0(M) = b_n(M) = 1$.

On the other hand, if (M,g) is a real Riemannian symmetric space of non-compact type, and of rank one then it is one of the four spaces: H^n, $\mathbf{C}H^n$, $\mathbf{H}H^n$, $\mathbf{O}H^2$, i.e., real hyperbolic space, complex hyperbolic space, quaternionic hyperbolic space and the octonionic hyperbolic plane (see [42]; [43]). We can normalize their Riemannian metrics so that the maximum of the sectional curvature is -1 (see also [42]; [43]). In this case, Dodziuk has proved that (M,g) carries no harmonic L^2-forms except when $r = n/2$ in which case $\mathbf{H}^r(M,\ \mathbb{R})$ is infinite dimensional (see [32]). This completes the proof of Theorem 5.

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REFERENCES

1. Bourguignon J.-P. Formules de Weitzenbock en dimension 4. *Seminare A. Besse sur la géometrie Riemannienne dimension 4, Cedic.* Ferman, Paris, 1981, vol. 3, pp. 308–333.
2. Petersen P. *Riemannian Geometry*. NY: Springer, 2006. 401 p.
3. Stepanov S.E. A class of closed forms and special Maxwell's equations. *Tensor N.S.*, 1997, vol. 58, no. 3, pp. 233–242.
4. Stepanov S.E. On conformal Killing 2-form of the electromagnetic field. *Journal of Geometry and Physics*, 2000, vol. 33, no. 3-4, pp. 191–209.
5. Tachibana S. On conformal Killing tensor in a Riemannian space. *Tohoku Mathematical Journal*, 1969, vol. 21, pp. 56–64.
6. Stepanov S.E., Tsyganok I.I., and Dmitrieva T.V. Harmonic and conformally Killing forms on complete Riemannian manifold. *Russian Mathematics*, 2017, vol. 61, no. 3, pp. 44–48.
7. Stepanov S.E. A new strong Laplacian on differential forms. *Mathematical Notes*, 2004, vol. 76, no. 3, pp. 420–425.
8. Stepanov S.E. and Mikesh J. Eigenvalues of the Tachibana operator which acts on differential forms. *Differential Geometry and its Applications*, 2014, vol. 35, pp. 19–25.
9. Frolov V.P. and Zelenkov A. *Introduction to Black Hole Physics*. Oxford: Oxford Univ. Press, 2011. 488 p.
10. Stephani H., Kramer D., Maccallum M., Hoenselaers C., and Herlt E. *Exact Solutions of Einstein's Field Equations*. Cambridge: Cambridge University Press, 2003. n.p.
11. Chen B.-Y. A simple characterization of generalized Robertson-Walker spacetimes, *General Relativity and Gravitation*, 2014, vol. 46, no. 11, p 5. DOI: 10.1007/s10714-014-1833-9.
12. Stepanov S. and Tsyganok I. Conformal Killing L^2-forms on complete Riemannian manifolds with nonpositive curvature operator. *Journal of Mathematical Analysis and Applications*, 2018, vol. 458, pp. 1–8.
13. Chavel I. On Riemannian symmetric spaces of rank one. *Advances in Mathematics*, 1997, vol. 4, pp. 236–263.
14. Koboyashi S. and Nomizu K. *Foundations of differential geometry*, vol. 1. New York-London: Int. Publishers, 1963. 345 p.

15. Baum H. and Juhl A. *Conformal differential geometry. Q-curvature and conformal holonomy*. Berlin, Birkhäuser Verlag AG, 2010. 152 p.
16. Stepanov S.E., Jukl M., and Mikeš J. *On dimensions of vector spaces of conformal Killing forms, Algebra, Geometry and Mathematical Physics*. Springer-Verlag, Berlin and Heidelberg, 2014, pp. 495–510.
17. Chavel I. *Riemannian Symmetric Spaces of Rank One (Lecture Notes in Pure and Applied Mathematics, vol. 5)*. New York, Marcel Dekker, 1972. 81 p.
18. Kashiwada T. On conformal Killing tensor. *Natural. Sci. Rep. Ochanomizu Univ.*, 1968, vol. 19, no 2, pp. 67–74.
19. Yano K. and Bochner S. *Curvature and Betti numbers*. Prinseton, Prinseton Univ. Press, 1953. 189 p.
20. Carter B. and Mc Lenaghan R.G. Generalized total angular momentum operator for the Dirac equation in curved space-time. *Phys. Rev. D*, 1979, vol. 19, pp. 1093–1097.
21. Gibbons G.W. and Ruback P.J. The hidden symmetries of multicentre metrics. *Commun. Math. Phys.*, 1988, pp. 115, pp. 267–300.
22. Charlton P. and Benn I.M. Dirac symmetry operators from conformal Killing-Yano tensor. *Classical Quantum Gravity*, 1997, vol. 14, no. 5, pp. 1037–1042.
23. Kubiznak D. Black hole spacetimes with Killing-Yano symmetries. *XVIth International Congress on Math. Physics*: Prague, Czech Republic, 3-8 August 2009, Word Scientific Publ. Co., Singapore, 2010, pp. 580–587.
24. Hodge W.V.D. *The theory and applications of harmonic integrals*. Cambridge, Cambridge Univ. Press, 1941. 281 p.
25. Hodge W.V.D. A Dirichlet problem for harmonic functions with applications to analytic varieties. *Proc. London Math. Soc*., 1934, vol. 36, pp. 257–303.
26. Helgason S. *Differential geometry and symmetric spaces.* USA, AMS Chelsea Publishing, 2001.
27. Stepanov S.E. and Mikeš J. Betti and Tachibana numbers of compact Riemannian manifolds. *Differential Geom. Appl*., 2013, vol. 31, no. 4, pp. 486–495.
28. Stepanov S.E. and Mikeš J. Betti and Tachibana numbers. *Miskolc Math. Notes*, 2013, vol. 14, no. 2, pp. 475–486.
29. Stepanov S.E. Curvature and Tachibana numbers. *Mat. Sb.*, 2011, vol. 202, no 7, pp. 135–146.
30. Tsagas G. On the Killing tensor fields on a compact Riemannian manifold. *Balkan Journal of Geometry and its Applications*, 1996, vol. 1, no. 1, pp. 1–97.
31. Stepanov S.E. and Tsyganok I.I. Conformal Killing L^2-forms on complete Riemannian manifolds with nonpositive curvature operator. *Journal Math. Anal. Appl*., 2018, vol. 458, pp. 1–8.
32. Dodziuk J. L^2 harmonic forms on rotationally symmetric Riemannian manifolds. *Proceedings of the American Mathematical Society*, 1979, vol. 77, no. 3, pp. 395–400.
33. Bérard P.H. From vanishing theorems to estimating theorems: the Bochner technique revisited. *Bulletin of the American Mathematical Society*, 1988, vol. 19, no. 2, pp. 371–406.
34. Li P. *Geometric Analysis*. Cambridge, Cambridge University Press, 2012.
35. Duchesne B. Infinite dimensional Riemannian symmetric spaces with fixed-sing curvature operator. *Ann. Inst. Fourier*, 2015, vol. 65, no. 1, pp. 211–244.
36. Yau S.T. Some function-theoretic properties of complete Riemannian manifold and their applications to geometry. *Indiana University Mathematical Journal*, 1976, vol. 25, no. 7, pp. 659–679.
37. Becce A. *Einstein manifolds*. Berlin-Heidelberg, Springer-Verlag, 1987.
38. Terras A. *Harmonic analysis on symmetric spaces – higher rank spaces, positive definite matrix space and generalizations*. New York, Springer Science and Business Media, 2015.
39. Kobayashi S. and Nomizu K. *Foundations of differential geometry, Vol. II*. New York-London, Interscience Publishers, 1969.
40. Greene R.E. and Wu H. Integral of subharmonic functions on manifolds of nonnegative curvature. *Inventiones Math*., 1974, vol. 27, pp. 265–298.
41. Hall B.C. The range of the heat operator. *Contemporary Mathematics*, 2006, vol. 398, pp. 203–232.
42. Mostow G.D. Strong rigidity of locally symmetric spaces. *Annals of Mathematics Studies*, No. 78, Princeton, Princeton University Press, 1973.
43. Parker J. Hyperbolic spaces. *Jyväskylä Lectures in Mathematics*, 2008, vol. 2.

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